Alternatively, conditions on the form of the function $g(x,t)$ for the resultant equation to conform to a known type also can be obtained from the foregoing equations as

$$g(x,t) = \frac{X_t + T_t[C_2(T)X - C_1(T)X^n]}{X_x + T_x[C_2(T)X - C_1(T)X^n]} \tag{7a}$$

$$g(x,t) = \frac{X_t - T_t[C_1(T)X^2 + C_2(T)X + C_3(T)]}{X_x - T_x[C_1(T)X^2 + C_2(T)X + C_3(T)]} \tag{7b}$$

$$g(x,t) = \frac{X_t - T_t[C_1(T)F(X)]}{X_x - T_x[C_1(T)F(X)]} \tag{7c}$$

$$g(x,t) = \frac{X_t - T_t[M(X,T)/N(X,T)]}{X_x - T_x[M(X,T)/N(X,T)]} \tag{7d}$$

Thus any first-order, nonlinear, nonautonomous system given by Eq. (1) along with one of the Eqs. (7a–7d) can be transformed to a classical equation (6a–6d) by the transformation laws (2) and (3).

Both the cases considered by Mason[1] [Eq. (1) and Eq. (7) of Ref. (1)] are but particular examples of Eq. (7a) and (7b) [along with Eq. (1)] with $X_x = f(x)$, $X_t = 0$, $T_t = 1$, and $T_x = 0$. Other cases of interest along with application of the technique to higher-order systems can be found in Refs. 2–5.

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"Catastrophic" Pressure Peaks in Solid-Propellant Combustion

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In two recent Technical Notes,[1,2] comments have been reported concerning the mechanism by which "catastroghic instability" develops in certain solid-propellant combustors. These Notes seek to explain experimental results in an earlier Note.[3] The claim is made that the mechanism arises from coupling of axial and transverse modes under the experimentally observed[3] conditions. It is my opinion that these contributions[1,2] are not original and not clearly relevant to the phenomenon they seek to explain.

As to originality, it should be noted that the earlier paper[3] reporting the phenomenon in question stated that catastrophic instability occurred when the frequencies of unstable transverse modes approached integral multiples of the frequencies of higher unstable axial modes. This statement implied recognition of the modes in question, and the frequency of the relevant transverse modes and axial modes were shown graphically, using the classical acoustic theory on which calculations in Ref. 1 and 2 are based. In this respect, the calculation of an expression relating mode frequencies in Ref. 1 contributes nothing new. Reference 1 goes on to suggest that the critical mode-frequency relationship under discussion may be conducive to some coupling between the modes, but does not propose a mechanism. In the original paper[3] it was noted that no such coupling is possible within the framework of linear analysis defining the modes under consideration, since the modes are orthogonal. One would presumably have to go outside the body of linear analysis to explain the coupling. Since this is not done in Refs. 1 and 2, it is difficult to see how a coupling mechanism can be claimed.

In the second of the recent Notes, the question of mode coupling is explored further, still using linearized analysis. The presence of mean flow or combustion is still neglected; excitation of modes is assumed to occur by a complicated independent vibration of the end surface of the circular cylindrical cavity (as contrasted to a coupled combustion vibration distributed over the side walls in the original experiment). In this analysis, it is shown that the coefficient of a particular term in the series solution becomes relatively large when the driving frequency approaches the mode frequency represented by that term. This observation is a simple demonstration of the classical concept of resonance, and establishes nothing about the severity of oscillations other than the point that the cavity impedance is low at the resonant frequency.† Following the demonstration of resonance in radial modes, the author again calculates the length to radius ratio of a circular cylinder required to give harmonically related frequencies for radial and axial modes, without comment on the purpose of the calculation. The calculations in no way demonstrate coupling between modes, and the boundary conditions used do not simulate those in the original combustion problem.

Aside from the foregoing questions, it is not clear that Refs. 1 and 2 examine the correct problem in the first place. The original paper[3] reported that, for the mode-frequency conditions under discussion, the solid-propellant combustor developed severe pressure peaks that were illustrated by a typical test record. It was stated specifically that oscillations were already present in transverse and axial modes before the pressure peaks developed, and that the peaks occurred when the frequencies of these unstable modes became harmonically related. It was stated that it could not yet be determined whether the oscillatory behavior became still more severe at that time, or whether the pressure peaks were due to an anomalous increase in quasi-steady burning rate under the harmonic frequency condition. This experimental question is still unresolved because of the problems of adequate separation of mode frequencies and accurate amplitude measurement at the high frequencies involved (10–60 k cps). It was noted in the original paper that the burning surface of the propellant became severely rippled during the high amplitude oscillatory behavior, a condition that could be conducive to and symptomatic of large anomalies in burning rate. The recent Notes[1,2] have directed comments to the question of severity of oscillations, a topic on which the original[3] or any other paper presents no information. In any case, it is not clear that one must seek mode coupling to explain the results, since the modes involved were already unstable.

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† No argument about energy transfer into the system is pursued and the author does not seem to recognize that destabilizing combustion oscillations tend to occur at mode frequencies because they are excited by gas oscillations at the frequencies of the most conservative modes.

In summary, the recent Notes[1,2] have 1) developed some frequency relations that were presented in the original paper, 2) proposed mode coupling arguments with an analytical model in which mode coupling is impossible, 3) rediscovered resonance and apparently confused concurrent resonance in two modes with coupling between modes, 4) in the process explored a model whose boundary conditions are completely different from the problem to which the analysis was ostensibly directed, and 5) has apparently confused the previously reported peaks in quasi-steady pressure with occurrence of increased amplitude of pressure oscillations.

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Reply by Author to E. W. Price

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IN the preceding Comment, Price has raised several objections to my recent Technical Notes[1,2] on large amplitude instability in rocket motors. His most important claims are that the Notes contribute nothing original to the literature and that they are not clearly relevant to the phenomenon I sought to study. It is my opinion that these claims are not valid, and that most of the objections raised are inapplicable.

Inasmuch as originality is questioned, it is appropriate to state here that the striking findings of Crump and Price[3] were fully acknowledged in both of my notes. In fact, the notes were motivated by their experimental results. However, one must differentiate between the findings of Crump and Price and those presented in Refs. 1 and 2. Crump and Price found experimentally that when the catastrophic pressure peaks occurred, the frequency of the transverse modes was an integer multiple of an unstable longitudinal mode. To show these results explicitly they plotted, in Fig. 4 of Ref. 3, the frequencies of the first two transverse modes of oscillation (as given by the well-known formula $f_{ms} = a_0\alpha_{ms}/2R$) vs the cavity radius. Of the many points shown on these plots, only four are experimental points; they give the frequencies at which catastrophic pressure peaks actually occurred. All of the other points (which I incorrectly identified as data points) merely show some frequencies at which the longitudinal and transverse modes would be related harmonically. With reference to the experimental points, Crump and Price stated that such striking results could be explained only by means of a nonlinear mechanism. They did not state why those specific integers of multiplicity were found, nor did they present any analysis showing that large pressure peaks would indeed occur under the experimentally found conditions.

In Ref. 2, I used a very simple analytical model to show that the empirical equation I had presented in Ref. 1 was indeed a condition for high-amplitude oscillations in cylindrical cavities. I considered there the simplest case where such large-amplitude oscillations would arise; namely, outgoing waves in a cylinder driven at one end by a piston oscillating (at a frequency ω) along the cylinder axis, and with a velocity that depended only on the radial coordinate. The case when there are reflected waves and where the piston velocity depends also on the angular coordinate gives basically the same result, i.e., the acoustic pressure is (except for a constant) given by

$$P\alpha \sum_{m,s} \frac{\cos(m\theta)\,\sin(\omega t) J_m(\pi\alpha_{ms}r/R)}{\sin k_x L[\omega^2 - (\pi\alpha_{ms}a_0/R)^2]^{1/2}} \cos k_x(x - L) \quad (1)$$

where $k_x^2 = (\omega/a_0)^2 - (\pi\alpha_{ms}/R)^2$. Clearly, large amplitudes of oscillation will result under some conditions of interest. The case $k_xL = N\pi$, $N = 1, 2, \ldots$ gives longitudinal resonance and leads to wave amplitudes that are limited either by dissipation effects or by shock-wave formation, and therefore cannot result in very large amplitude oscillations. The other possibility for large-amplitude waves is provided by cutoff conditions, i.e., if the piston frequency is $\omega = \pi\alpha_{ms}a_0/R$. (For a given frequency, this condition may occur if R varies, such as in a radially burning solid-propellant rocket motor.) At this frequency, the oscillations will be purely transverse ($k_x \to 0$). One can, in fact, say that the transverse oscillations grow at the expense of the longitudinal modes. Also, the amplitudes of the transverse waves would, at this frequency, be very large since, as Maslen and Moore[4] have shown, the transverse waves are not limited by shock formation as in the longitudinal-wave case. One would therefore expect very large pressure peaks if the foregoing condition is satisfied. In particular, if the piston frequency is given by $\omega = n\pi a_0/L$, n an integer (i.e., the nth resonant frequency of a cavity of length L with no transverse waves) one finds that the condition for large amplitudes reduces to $n = \alpha_{ms}(L/R)$ (Eq. 1 of Ref. 2). In fact, if one substitutes in this equation the values of L and R from the experimental catastrophic pressure peaks found by Crump and Price, one finds exact agreement between the calculated value of n and the integers of multiplicity found experimentally. Consider, for example, test number 690 (Fig. 4 of Ref. 3). The value of the cavity diameter at the instant when large pressure peaks occurred in the first transverse mode was found to be approximately equal to 0.64 in., and the (fixed) length was 10 in. With these values of R and L, and with $\alpha_{10} = 0.586$, one finds $n = 18$; this is the value found experimentally by Crump and Price. Similar results are found for the other experimental points. Surely, one cannot discard as irrelevant analytical predictions that agree with experimental observations.

Price also has raised several objections to the model I considered in Ref. 2, and to my interpretation of the phenomena involved. The model, in fact, does not take into account burning of the propellant, mean gas flow, and other effects that were present in the actual experiments, but considers, as done by other authors, oscillations in a "cold" cavity. This approach is not entirely unrealistic since the cavity of a rocket motor acts as an acoustic resonator with unstable modes of oscillation identically equal to those predicted by acoustic theory for cold cavities. In the actual case, these modes are excited by the energy released during burning. The manner in which this energy transfer takes place is a most serious problem and is not considered in my Notes. However, the characteristic modes of oscillation of a cavity can be excited by many other means, such as a piston (as done in my Note), a gas jet, heat addition, etc. The model used in my Note is, therefore, a useful one since it provides excitation of acoustic modes in a cylindrical cavity with very simple mathematical arguments. Clearly, such simple analysis cannot answer all important questions that arise with regard to the actual experiments of Crump and Price. An example is provided by the fact that the analysis, although predicting very large amplitudes of oscillation under certain conditions, does not predict a net buildup of the mean

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